



## LIMIT AND CONTINUITY

Consider the function  $f(x) = \frac{x^2 - 1}{x - 1}$

You can see that the function  $f(x)$  is not defined at  $x = 1$  as  $x - 1$  is in the denominator. Take the value of  $x$  very nearly equal to but not equal to 1 as given in the tables below. In this case  $x - 1 \neq 0$  as  $x \neq 1$ .

$\therefore$  We can write  $f(x) = \frac{x^2 - 1}{x - 1} = \frac{(x + 1)(x - 1)}{(x - 1)} = x + 1$ , because  $x - 1 \neq 0$  and so division by

$(x - 1)$  is possible.

Table - 1

x	f(x)
0.5	1.5
0.6	1.6
0.7	1.7
0.8	1.8
0.9	1.9
0.91	1.91
:	:
:	:
0.99	1.99
:	:
:	:
0.9999	1.9999

Table - 2

x	f(x)
1.9	2.9
1.8	2.8
1.7	2.7
1.6	2.6
1.5	2.5
:	:
:	:
1.1	2.1
1.01	2.01
1.001	2.001
:	:
:	:
1.00001	2.00001

In the above tables, you can see that as  $x$  gets closer to 1, the corresponding value of  $f(x)$  also gets closer to 2.

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However, in this case  $f(x)$  is not defined at  $x = 1$ . The idea can be expressed by saying that the limiting value of  $f(x)$  is 2 when  $x$  approaches to 1.

Let us consider another function  $f(x) = 2x$ . Here, we are interested to see its behavior near the point 1 and at  $x = 1$ . We find that as  $x$  gets nearer to 1, the corresponding value of  $f(x)$  gets closer to 2 at  $x = 1$  and the value of  $f(x)$  is also 2.

So from the above findings, what more can we say about the behaviour of the function near  $x = 2$  and at  $x = 2$  ?

In this lesson we propose to study the behaviour of a function near and at a particular point where the function may or may not be defined.


**OBJECTIVES**

After studying this lesson, you will be able to :

- illustrate the notion of limit of a function through graphs and examples;
- define and illustrate the left and right hand limits of a function  $y = f(x)$  at  $x = a$ ;
- define limit of a function  $y = f(x)$  at  $x = a$ ;
- state and use the basic theorems on limits;
- establish the following on limits and apply the same to solve problems :

$$(i) \quad \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1} \quad (x \neq a)$$

$$(ii) \quad \lim_{x \rightarrow 0} \sin x = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} \cos x = 1$$

$$(iii) \quad \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \qquad (iv) \quad \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e$$

$$(vi) \quad \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1 \qquad (vii) \quad \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

- define and interpret geometrically the continuity of a function at a point;
- define the continuity of a function in an interval;
- determine the continuity or otherwise of a function at a point; and
- state and use the theorems on continuity of functions with the help of examples.

**EXPECTED BACKGROUND KNOWLEDGE**

- Concept of a function
- Drawing the graph of a function
- Concept of trigonometric function
- Concepts of exponential and logarithmic functions

**20.1 LIMIT OF A FUNCTION**

In the introduction, we considered the function  $f(x) = \frac{x^2 - 1}{x - 1}$ . We have seen that as  $x$  approaches 1,  $f(x)$  approaches 2. In general, if a function  $f(x)$  approaches  $L$  when  $x$  approaches 'a', we say that  $L$  is the limiting value of  $f(x)$

Symbolically it is written as

$$\lim_{x \rightarrow a} f(x) = L$$

Now let us find the limiting value of the function  $(5x - 3)$  when  $x$  approaches 0.

i.e. 
$$\lim_{x \rightarrow 0} (5x - 3)$$

For finding this limit, we assign values to  $x$  from left and also from right of 0.

x	-0.1	-0.01	-0.001	-0.0001.....
5x - 3	-3.5	-3.05	-3.005	-3.0005.....

x	0.1	0.01	0.001	0.0001.....
5x - 3	-2.5	-2.95	-2.995	-2.9995.....

It is clear from the above that the limit of  $(5x - 3)$  as  $x \rightarrow 0$  is -3

i.e., 
$$\lim_{x \rightarrow 0} (5x - 3) = -3$$

This is illustrated graphically in the Fig. 20.1

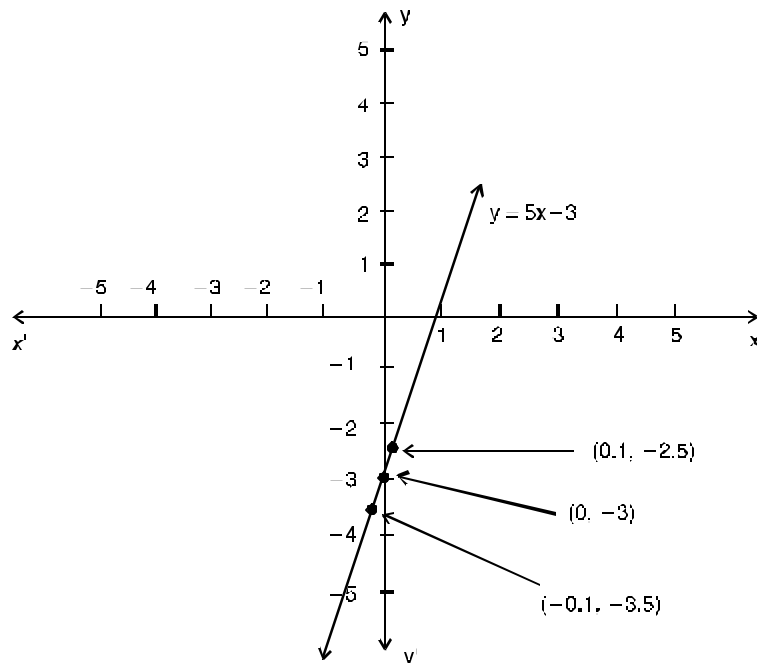


Fig. 20.1



Notes

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The method of finding limiting values of a function at a given point by putting the values of the variable very close to that point may not always be convenient.

We, therefore, need other methods for calculating the limits of a function as  $x$  (independent variable) tends to a finite quantity, say  $a$

**Consider an example :** Find  $\lim_{x \rightarrow 3} f(x)$ , where  $f(x) = \frac{x^2 - 9}{x - 3}$

We can solve it by the method of substitution. Steps of which are as follows :

<p><b>Step 1:</b> We consider a value of <math>x</math> close to <math>a</math> say <math>x = a + h</math>, where <math>h</math> is a very small positive number. Clearly, as <math>x \rightarrow a</math>, <math>h \rightarrow 0</math></p>	<p>For <math>f(x) = \frac{x^2 - 9}{x - 3}</math> we write <math>x = 3 + h</math>, so that as <math>x \rightarrow 3, h \rightarrow 0</math></p>
<p><b>Step 2 :</b> Simplify <math>f(x) = f(a + h)</math></p>	<p>Now <math>f(x) = f(3 + h)</math></p> $= \frac{(3 + h)^2 - 9}{3 + h - 3}$ $= \frac{h^2 + 6h}{h}$ $= h + 6$
<p><b>Step 3 :</b> Put <math>h = 0</math> and get the required result</p>	<p><math>\therefore \lim_{x \rightarrow 3} f(x) = \lim_{h \rightarrow 0} (6 + h)</math></p> <p>As <math>x \rightarrow 0, h \rightarrow 0</math></p> <p>Thus, <math>\lim_{x \rightarrow 3} f(x) = 6 + 0 = 6</math></p> <p>by putting <math>h = 0</math>.</p>

**Remarks :** It may be noted that  $f(3)$  is not defined, however, in this case the limit of the function  $f(x)$  as  $x \rightarrow 3$  is 6.

Now we shall discuss other methods of finding limits of different types of functions.

**Consider the example :**

Find  $\lim_{x \rightarrow 1} f(x)$ , where  $f(x) = \begin{cases} \frac{x^3 - 1}{x^2 - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$

Here, for  $x \neq 1, f(x) = \frac{x^3 - 1}{x^2 - 1}$

$$= \frac{(x - 1)(x^2 + x + 1)}{(x - 1)(x + 1)}$$



Notes

It shows that if  $f(x)$  is of the form  $\frac{g(x)}{h(x)}$ , then we may be able to solve it by the method of factors. In such case, we follow the following steps :

<p><b>Step 1.</b> Factorise <math>g(x)</math> and <math>h(x)</math></p>	<p><b>Sol.</b></p> $f(x) = \frac{x^3 - 1}{x^2 - 1}$ $= \frac{(x-1)(x^2 + x + 1)}{(x-1)(x+1)}$ <p>(<math>\because x \neq 1, \therefore x - 1 \neq 0</math> and as such can be cancelled)</p>
<p><b>Step 2 :</b> Simplify <math>f(x)</math></p>	<p><math>\therefore f(x) = \frac{x^2 + x + 1}{x + 1}</math></p>
<p><b>Step 3 :</b> Putting the value of <math>x</math>, we get the required limit.</p>	<p><math>\therefore \lim_{x \rightarrow 1} \frac{x^3 - 1}{x^2 - 1} = \frac{1 + 1 + 1}{1 + 1} = \frac{3}{2}</math></p> <p>Also <math>f(1) = 1</math> (given)</p> <p>In this case, <math>\lim_{x \rightarrow 1} f(x) \neq f(1)</math></p>

Thus, the limit of a function  $f(x)$  as  $x \rightarrow a$  may be different from the value of the function at  $x = a$ .

Now, we take an example which cannot be solved by the method of substitutions or method of factors.

Evaluate  $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$

Here, we do the following steps :

**Step 1.** Rationalise the factor containing square root.

**Step 2.** Simplify.

**Step 3.** Put the value of  $x$  and get the required result.

**Solution :**

$$\frac{\sqrt{1+x} - \sqrt{1-x}}{x} = \frac{(\sqrt{1+x} - \sqrt{1-x})(\sqrt{1+x} + \sqrt{1-x})}{x(\sqrt{1+x} + \sqrt{1-x})}$$

$$= \frac{\sqrt{(1+x)^2} - \sqrt{(1-x)^2}}{x(\sqrt{1+x} + \sqrt{1-x})}$$

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$$\begin{aligned}
 &= \frac{(1+x) - (1-x)}{x(\sqrt{1+x} + \sqrt{1-x})} \\
 &= \frac{1+x-1+x}{x(\sqrt{1+x} + \sqrt{1-x})} \\
 &= \frac{2x}{x(\sqrt{1+x} + \sqrt{1-x})} \quad [ \because x \neq 0, \therefore \text{It can be cancelled} ] \\
 &= \frac{2}{\sqrt{1+x} + \sqrt{1-x}} \\
 \therefore \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} &= \lim_{x \rightarrow 0} \frac{2}{\sqrt{1+x} + \sqrt{1-x}} \\
 &= \frac{2}{\sqrt{1+0} + \sqrt{1-0}} \\
 &= \frac{2}{1+1} = 1
 \end{aligned}$$

**20.2 LEFT AND RIGHT HAND LIMITS**

You have already seen that  $x \rightarrow a$  means  $x$  takes values which are very close to 'a', i.e. either the value is greater than 'a' or less than 'a'.

In case  $x$  takes only those values which are less than 'a' and very close to 'a' then we say  $x$  is approaching 'a' from the left and we write it as  $x \rightarrow a^-$ . Similarly, if  $x$  takes values which are greater than 'a' and very close to 'a' then we say  $x$  is approaching 'a' from the right and we write it as  $x \rightarrow a^+$ .

Thus, if a function  $f(x)$  approaches a limit  $\ell_1$ , as  $x$  approaches 'a' from left, we say that the left hand limit of  $f(x)$  as  $x \rightarrow a$  is  $\ell_1$ .

We denote it by writing

$$\lim_{x \rightarrow a^-} f(x) = \ell_1 \quad \text{or} \quad \lim_{h \rightarrow 0} f(a-h) = \ell_1, h > 0$$

Similarly, if  $f(x)$  approaches the limit  $\ell_2$ , as  $x$  approaches 'a' from right we say, that the right hand limit of  $f(x)$  as  $x \rightarrow a$  is  $\ell_2$ .

We denote it by writing

$$\lim_{x \rightarrow a^+} f(x) = \ell_2 \quad \text{or} \quad \lim_{h \rightarrow 0} f(a+h) = \ell_2, h > 0$$

**Working Rules**

Finding the right hand limit i.e.,

$$\lim_{x \rightarrow a^+} f(x)$$

Finding the left hand limit, i.e.,

$$\lim_{x \rightarrow a^-} f(x)$$

Put  $x = a + h$

Find  $\lim_{h \rightarrow 0} f(a + h)$

Put  $x = a - h$

Find  $\lim_{h \rightarrow 0} f(a - h)$



**Note :** In both cases remember that  $h$  takes only positive values.

**20.3 LIMIT OF FUNCTION  $y = f(x)$  AT  $x = a$**

**Consider an example :**

Find  $\lim_{x \rightarrow 1} f(x)$ , where  $f(x) = x^2 + 5x + 3$

Here 
$$\begin{aligned} \lim_{x \rightarrow 1^+} f(x) &= \lim_{h \rightarrow 0} [(1 + h)^2 + 5(1 + h) + 3] \\ &= \lim_{h \rightarrow 0} [1 + 2h + h^2 + 5 + 5h + 3] \\ &= 1 + 5 + 3 = 9 \end{aligned} \quad \dots(i)$$

and 
$$\begin{aligned} \lim_{x \rightarrow 1^-} f(x) &= \lim_{h \rightarrow 0} [(1 - h)^2 + 5(1 - h) + 3] \\ &= \lim_{h \rightarrow 0} [1 - 2h + h^2 + 5 - 5h + 3] \\ &= 1 + 5 + 3 = 9 \end{aligned} \quad \dots(ii)$$

From (i) and (ii),  $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x)$

**Now consider another example :**

Evaluate : 
$$\lim_{x \rightarrow 3} \frac{|x - 3|}{x - 3}$$

Here 
$$\begin{aligned} \lim_{x \rightarrow 3^+} \frac{|x - 3|}{x - 3} &= \lim_{h \rightarrow 0} \frac{|(3 + h) - 3|}{(3 + h) - 3} \\ &= \lim_{h \rightarrow 0} \frac{|h|}{h} \\ &= \lim_{h \rightarrow 0} \frac{h}{h} \quad (\text{as } h > 0, \text{ so } |h| = h) \\ &= 1 \end{aligned} \quad \dots(iii)$$

and 
$$\begin{aligned} \lim_{x \rightarrow 3^-} \frac{|x - 3|}{x - 3} &= \lim_{h \rightarrow 0} \frac{|(3 - h) - 3|}{(3 - h) - 3} \\ &= \lim_{h \rightarrow 0} \frac{|-h|}{-h} \end{aligned}$$

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$$= \lim_{h \rightarrow 0} \frac{h}{-h} \quad (\text{as } h > 0, \text{ so } |-h| = h)$$

$$= -1 \quad \dots(\text{iv})$$

∴ From (iii) and (iv),  $\lim_{x \rightarrow 3^+} \frac{|x-3|}{x-3} \neq \lim_{x \rightarrow 3^-} \frac{|x-3|}{x-3}$

Thus, in the first example right hand limit = left hand limit whereas in the second example right hand limit  $\neq$  left hand limit.

Hence the left hand and the right hand limits may not always be equal.

We may conclude that

$\lim_{x \rightarrow 1} (x^2 + 5x + 3)$  exists (which is equal to 9) and  $\lim_{x \rightarrow 3} \frac{|x-3|}{x-3}$  does not exist.

**Note :**

$$\left. \begin{array}{l} \text{I} \quad \lim_{x \rightarrow a^+} f(x) = \ell \\ \text{and} \quad \lim_{x \rightarrow a^-} f(x) = \ell \end{array} \right\} \Rightarrow \lim_{x \rightarrow a} f(x) = \ell$$

$$\left. \begin{array}{l} \text{II} \quad \lim_{x \rightarrow a^+} f(x) = \ell_1 \\ \text{and} \quad \lim_{x \rightarrow a^-} f(x) = \ell_2 \end{array} \right\} \Rightarrow \lim_{x \rightarrow a} f(x) \text{ does not exist.}$$

$$\text{III} \quad \lim_{x \rightarrow a^+} f(x) \text{ or } \lim_{x \rightarrow a^-} f(x) \text{ does not exist} \Rightarrow \lim_{x \rightarrow a} f(x) \text{ does not exist.}$$

**20.4 BASIC THEOREMS ON LIMITS**

1.  $\lim_{x \rightarrow a} cx = c \lim_{x \rightarrow a} x$ , c being a constant.

To verify this, consider the function  $f(x) = 5x$ .

We observe that in  $\lim_{x \rightarrow 2} 5x$ , 5 being a constant is not affected by the limit.

$$\therefore \lim_{x \rightarrow 2} 5x = 5 \lim_{x \rightarrow 2} x = 5 \times 2 = 10$$

2.  $\lim_{x \rightarrow a} [g(x) + h(x) + p(x) + \dots] = \lim_{x \rightarrow a} g(x) + \lim_{x \rightarrow a} h(x) + \lim_{x \rightarrow a} p(x) + \dots$

where  $g(x), h(x), p(x), \dots$  are any function.

3.  $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$





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To verify this, consider  $f(x) = 5x^2 + 2x + 3$

and  $g(x) = x + 2$ .

Then  $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (5x^2 + 2x + 3)$

$$= 5 \lim_{x \rightarrow 0} x^2 + 2 \lim_{x \rightarrow 0} x + 3 = 3$$

$$\lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} (x + 2) = \lim_{x \rightarrow 0} x + 2 = 2$$

$$\therefore \lim_{x \rightarrow 0} (5x^2 + 2x + 3) \lim_{x \rightarrow 0} (x + 2) = 6 \quad \dots(i)$$

Again  $\lim_{x \rightarrow 0} [f(x) \cdot g(x)] = \lim_{x \rightarrow 0} [(5x^2 + 2x + 3)(x + 2)]$

$$= \lim_{x \rightarrow 0} (5x^3 + 12x^2 + 7x + 6)$$

$$= 5 \lim_{x \rightarrow 0} x^3 + 12 \lim_{x \rightarrow 0} x^2 + 7 \lim_{x \rightarrow 0} x + 6$$

$$= 6 \quad \dots(ii)$$

From (i) and (ii),  $\lim_{x \rightarrow 0} [(5x^2 + 2x + 3)(x + 2)] = \lim_{x \rightarrow 0} (5x^2 + 2x + 3) \lim_{x \rightarrow 0} (x + 2)$

$$4. \lim_{x \rightarrow a} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad \text{provided } \lim_{x \rightarrow a} g(x) \neq 0$$

To verify this, consider the function  $f(x) = \frac{x^2 + 5x + 6}{x + 2}$

$$\text{we have } \lim_{x \rightarrow -1} (x^2 + 5x + 6) = (-1)^2 + 5(-1) + 6$$

$$= 1 - 5 + 6$$

$$= 2$$

$$\text{and } \lim_{x \rightarrow -1} (x + 2) = -1 + 2$$

$$= 1$$

$$\frac{\lim_{x \rightarrow -1} (x^2 + 5x + 6)}{\lim_{x \rightarrow -1} (x + 2)} = \frac{2}{1} = 2 \quad \dots(i)$$

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Also

$$\lim_{x \rightarrow -1} \frac{(x^2 + 5x + 6)}{x + 2} = \lim_{x \rightarrow -1} \frac{(x + 3)(x + 2)}{x + 2} \left[ \begin{array}{l} \because x^2 + 5x + 6 \\ = x^2 + 3x + 2x + 6 \\ = x(x + 3) + 2(x + 3) \\ = (x + 3)(x + 2) \end{array} \right]$$

$$= \lim_{x \rightarrow -1} (x + 3)$$

$$= -1 + 3 = 2 \quad \dots\text{(ii)}$$

\(\therefore\) From (i) and (ii),

$$\lim_{x \rightarrow -1} \frac{x^2 + 5x + 6}{x + 2} = \frac{\lim_{x \rightarrow -1} (x^2 + 5x + 6)}{\lim_{x \rightarrow -1} (x + 2)}$$

We have seen above that there are many ways that two given functions may be combined to form a new function. The limit of the combined function as  $x \rightarrow a$  can be calculated from the limits of the given functions. To sum up, we state below some basic results on limits, which can be used to find the limit of the functions combined with basic operations.

If  $\lim_{x \rightarrow a} f(x) = \ell$  and  $\lim_{x \rightarrow a} g(x) = m$ , then

- (i)  $\lim_{x \rightarrow a} kf(x) = k \lim_{x \rightarrow a} f(x) = k\ell$  where  $k$  is a constant.
- (ii)  $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = \ell \pm m$
- (iii)  $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = \ell \cdot m$
- (iv)  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{\ell}{m}$ , provided  $\lim_{x \rightarrow a} g(x) \neq 0$

The above results can be easily extended in case of more than two functions.

**Example 20.1** Find  $\lim_{x \rightarrow 1} f(x)$ , where

$$f(x) = \begin{cases} \frac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$$

**Solution :**

$$f(x) = \frac{x^2 - 1}{x - 1}$$

$$= \frac{(x - 1)(x + 1)}{x - 1}$$



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$$= (x + 1) \quad [\because x \neq 1]$$

$$\therefore \lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} (x + 1)$$

$$= 1 + 1 = 2$$

**Note :**  $\frac{x^2 - 1}{x - 1}$  is not defined at  $x = 1$ . The value of  $\lim_{x \rightarrow 1} f(x)$  is independent of the value of  $f(x)$  at  $x = 1$ .

**Example 20.2** Evaluate :  $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2}$ .

**Solution :**

$$\lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2}$$

$$= \lim_{x \rightarrow 2} \frac{(x - 2)(x^2 + 2x + 4)}{(x - 2)}$$

$$= \lim_{x \rightarrow 2} (x^2 + 2x + 4) \quad [\because x \neq 2]$$

$$= 2^2 + 2 \times 2 + 4$$

$$= 12$$

**Example 20.3** Evaluate :  $\lim_{x \rightarrow 2} \frac{\sqrt{3-x} - 1}{2-x}$ .

**Solution :** Rationalizing the numerator, we have

$$\frac{\sqrt{3-x} - 1}{2-x} = \frac{\sqrt{3-x} - 1}{2-x} \times \frac{\sqrt{3-x} + 1}{\sqrt{3-x} + 1}$$

$$= \frac{3-x-1}{(2-x)(\sqrt{3-x} + 1)}$$

$$= \frac{2-x}{(2-x)(\sqrt{3-x} + 1)}$$

$$\therefore \lim_{x \rightarrow 2} \frac{\sqrt{3-x} - 1}{2-x} = \lim_{x \rightarrow 2} \frac{2-x}{(2-x)(\sqrt{3-x} + 1)}$$

$$= \lim_{x \rightarrow 2} \frac{1}{(\sqrt{3-x} + 1)} \quad [\because x \neq 2]$$

$$= \frac{1}{(\sqrt{3-2} + 1)}$$

$$= \frac{1}{1+1} = \frac{1}{2}$$

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**Example 20.4** Evaluate :  $\lim_{x \rightarrow 3} \frac{\sqrt{12-x} - x}{\sqrt{6+x} - 3}$ .

**Solution :** Rationalizing the numerator as well as the denominator, we get

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{\sqrt{12-x} - x}{\sqrt{6+x} - 3} &= \lim_{x \rightarrow 3} \frac{(\sqrt{12-x} - x)(\sqrt{12-x} + x)(\sqrt{6+x} + 3)}{\sqrt{6+x} - 3(\sqrt{6+x} + 3)(\sqrt{12-x} + x)} \\ &= \lim_{x \rightarrow 3} \frac{(12-x-x^2)}{6+x-9} \cdot \lim_{x \rightarrow 3} \frac{\sqrt{6+x} + 3}{\sqrt{12-x} + x} \\ &= \lim_{x \rightarrow 3} \frac{-(x+4)(x-3)}{(x-3)} \cdot \lim_{x \rightarrow 3} \frac{\sqrt{6+x} + 3}{\sqrt{12-x} + x} \quad [ \because x \neq 3 ] \\ &= -(3+4) \frac{6}{6} = -7 \end{aligned}$$

**Note :** Whenever in a function, the limits of both numerator and denominator are zero, you should simplify it in such a manner that the denominator of the resulting function is not zero. However, if the limit of the denominator is 0 and the limit of the numerator is non zero, then the limit of the function does not exist.

Let us consider the example given below :

**Example 20.5** Find  $\lim_{x \rightarrow 0} \frac{1}{x}$ , if it exists.

**Solution :** We choose values of  $x$  that approach 0 from both the sides and tabulate the corresponding values of  $\frac{1}{x}$ .

$x$	-0.1	-0.01	-0.001	-0.0001
$\frac{1}{x}$	-10	-100	-1000	-10000

$x$	0.1	.01	.001	.0001
$\frac{1}{x}$	10	100	1000	10000

We see that as  $x \rightarrow 0$ , the corresponding values of  $\frac{1}{x}$  are not getting close to any number.

Hence,  $\lim_{x \rightarrow 0} \frac{1}{x}$  does not exist. This is illustrated by the graph in Fig. 20.2



Notes

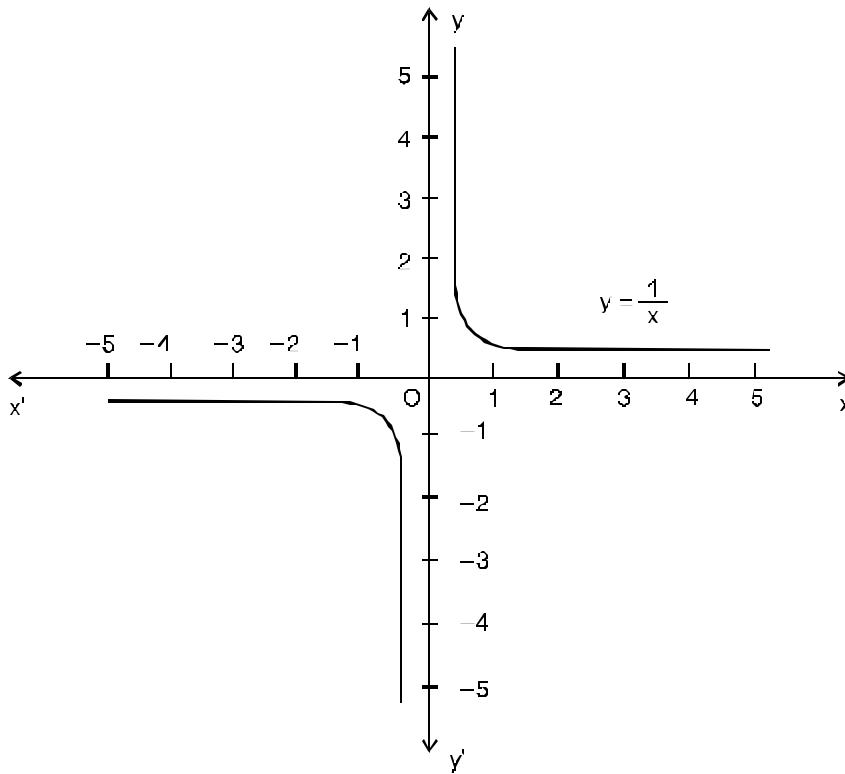


Fig. 20.2

**Example 20.6** Evaluate :  $\lim_{x \rightarrow 0} (|x| + |-x|)$

**Solution :** Since  $|x|$  has different values for  $x \geq 0$  and  $x < 0$ , therefore we have to find out both left hand and right hand limits.

$$\begin{aligned} \lim_{x \rightarrow 0^-} (|x| + |-x|) &= \lim_{h \rightarrow 0} (|0 - h| + |(0 - h)|) \\ &= \lim_{h \rightarrow 0} (|-h| + |-(h)|) \\ &= \lim_{h \rightarrow 0} h + h = \lim_{h \rightarrow 0} 2h = 0 \end{aligned} \quad \dots(i)$$

and

$$\begin{aligned} \lim_{x \rightarrow 0^+} (|x| + |-x|) &= \lim_{h \rightarrow 0} (|0 + h| + |(0 + h)|) \\ &= \lim_{h \rightarrow 0} h + h = \lim_{h \rightarrow 0} 2h = 0 \end{aligned} \quad \dots(ii)$$

From (i) and (ii),

$$\lim_{x \rightarrow 0} (|x| + |-x|) = \lim_{h \rightarrow 0^+} [|x| + |-x|]$$

Thus ,  $\lim_{h \rightarrow 0} [|x| + |-x|] = 0$

**Note :** We should remember that left hand and right hand limits are specially used when (a) the functions under consideration involve modulus function, and (b) function is defined by more than one rule.

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**Example 20.7** Find the value of 'a' so that

$$\lim_{x \rightarrow 1} f(x) \text{ exist, where } f(x) = \begin{cases} 3x + 5, & x \leq 1 \\ 2x + a, & x > 1 \end{cases}$$

**Solution :**

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (3x + 5) \quad [ \because f(x) = 3x + 5 \text{ for } x \leq 1 ]$$

$$= \lim_{h \rightarrow 0} [3(1-h) + 5]$$

$$= 3 + 5 = 8 \quad \dots(i)$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (2x + a) \quad [ \because f(x) = 2x + a \text{ for } x > 1 ]$$

$$= \lim_{h \rightarrow 0} (2(1+h) + a)$$

$$= 2 + a \quad \dots(ii)$$

We are given that  $\lim_{x \rightarrow 1} f(x)$  will exist provided

$$\Rightarrow \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x)$$

 $\therefore$  From (i) and (ii),

$$2 + a = 8$$

$$\therefore \text{or, } a = 6$$

**Example 20.8** If a function  $f(x)$  is defined as

$$f(x) = \begin{cases} x, & 0 \leq x < \frac{1}{2} \\ 0, & x = \frac{1}{2} \\ x-1, & \frac{1}{2} < x \leq 1 \end{cases}$$

Examine the existence of  $\lim_{x \rightarrow \frac{1}{2}} f(x)$ .**Solution :** Here

$$f(x) = \begin{cases} x, & 0 \leq x < \frac{1}{2} \end{cases} \quad \dots(i)$$

$$f(x) = \begin{cases} 0, & x = \frac{1}{2} \\ x-1, & \frac{1}{2} < x \leq 1 \end{cases} \quad \dots(ii)$$

$$\lim_{x \rightarrow \left(\frac{1}{2}\right)^-} f(x) = \lim_{h \rightarrow 0} f\left(\frac{1}{2} - h\right)$$

$$= \lim_{x \rightarrow 0} \left(\frac{1}{2} - h\right) \quad [ \because \frac{1}{2} - h < \frac{1}{2} \text{ and from (i), } f\left(\frac{1}{2} - h\right) = \frac{1}{2} - h ]$$



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$$= \frac{1}{2} - 0 = \frac{1}{2} \quad \dots\text{(iii)}$$

$$\lim_{x \rightarrow \left(\frac{1}{2}\right)^+} f(x) = \lim_{h \rightarrow 0} f\left(\frac{1}{2} + h\right)$$

$$= \lim_{h \rightarrow 0} \left[ \left(\frac{1}{2} + h\right) - 1 \right] \left[ \because \frac{1}{2} + h > \frac{1}{2} \text{ and from (ii), } f\left(\frac{1}{2} + h\right) = \left(\frac{1}{2} + h\right) - 1 \right]$$

$$= \frac{1}{2} + -1$$

$$= -\frac{1}{2} \quad \dots\text{(iv)}$$

From (iii) and (iv), left hand limit  $\neq$  right hand limit

$\therefore \lim_{x \rightarrow \frac{1}{2}} f(x)$  does not exist.



### CHECK YOUR PROGRESS 20.1

1. Evaluate each of the following limits :

(a)  $\lim_{x \rightarrow 2} [2(x+3)+7]$

(b)  $\lim_{x \rightarrow 0} (x^2 + 3x + 7)$

(c)  $\lim_{x \rightarrow 1} [(x+3)^2 - 16]$

(d)  $\lim_{x \rightarrow 1} [(x+1)^2 + 2]$

(e)  $\lim_{x \rightarrow 0} [(2x+1)^3 - 5]$

(f)  $\lim_{x \rightarrow 1} (3x+1)(x+1)$

2. Find the limits of each of the following functions :

(a)  $\lim_{x \rightarrow 5} \frac{x-5}{x+2}$

(b)  $\lim_{x \rightarrow 1} \frac{x+2}{x+1}$

(c)  $\lim_{x \rightarrow 1} \frac{3x+5}{x-10}$

(d)  $\lim_{x \rightarrow 0} \frac{px+q}{ax+b}$

(e)  $\lim_{x \rightarrow 3} \frac{x^2-9}{x-3}$

(f)  $\lim_{x \rightarrow 5} \frac{x^2-25}{x+5}$

(g)  $\lim_{x \rightarrow 2} \frac{x^2-x-2}{x^2-3x+2}$

(h)  $\lim_{x \rightarrow \frac{1}{3}} \frac{9x^2-1}{3x-1}$

3. Evaluate each of the following limits:

(a)  $\lim_{x \rightarrow 1} \frac{x^3-1}{x-1}$

(b)  $\lim_{x \rightarrow 0} \frac{x^3+7x}{x^2+2x}$

(c)  $\lim_{x \rightarrow 1} \frac{x^4-1}{x-1}$

(d)  $\lim_{x \rightarrow 1} \left[ \frac{1}{x-1} - \frac{2}{x^2-1} \right]$

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4. Evaluate each of the following limits :

(a)  $\lim_{x \rightarrow 0} \frac{\sqrt{4+x} - \sqrt{4-x}}{x}$

(b)  $\lim_{x \rightarrow 0} \frac{\sqrt{2+x} - \sqrt{2}}{x}$

(c)  $\lim_{x \rightarrow 3} \frac{\sqrt{3+x} - \sqrt{6}}{x-3}$

(d)  $\lim_{x \rightarrow 0} \frac{x}{\sqrt{1+x} - 1}$

(e)  $\lim_{x \rightarrow 2} \frac{\sqrt{3x-2} - x}{2 - \sqrt{6-x}}$

5. (a) Find  $\lim_{x \rightarrow 0} \frac{2}{x}$ , if it exists. (b) Find  $\lim_{x \rightarrow 2} \frac{1}{x-2}$ , if it exists.

6. Find the values of the limits given below :

(a)  $\lim_{x \rightarrow 0} \frac{x}{5 - |x|}$

(b)  $\lim_{x \rightarrow 2} \frac{1}{|x+2|}$

(c)  $\lim_{x \rightarrow 2} \frac{1}{|x-2|}$

(d) Show that  $\lim_{x \rightarrow 5} \frac{|x-5|}{x-5}$  does not exist.

7. (a) Find the left hand and right hand limits of the function

$$f(x) = \begin{cases} -2x + 3, & x \leq 1 \\ 3x - 5, & x > 1 \end{cases} \text{ as } x \rightarrow 1$$

(b) If  $f(x) = \begin{cases} x^2, & x \leq 1 \\ 1, & x > 1 \end{cases}$ , find  $\lim_{x \rightarrow 1} f(x)$ (c) Find  $\lim_{x \rightarrow 4} f(x)$  if it exists, given that  $f(x) = \begin{cases} 4x + 3, & x < 4 \\ 3x + 7, & x \geq 4 \end{cases}$ 8. Find the value of 'a' such that  $\lim_{x \rightarrow 2} f(x)$  exists, where  $f(x) = \begin{cases} ax + 5, & x < 2 \\ x - 1, & x \geq 2 \end{cases}$ 9. Let  $f(x) = \begin{cases} x, & x < 1 \\ 1, & x = 1 \\ x^2, & x > 1 \end{cases}$ Establish the existence of  $\lim_{x \rightarrow 1} f(x)$ .10. Find  $\lim_{x \rightarrow 2} f(x)$  if it exists, where

$$f(x) = \begin{cases} x - 1, & x < 2 \\ 1, & x = 2 \\ x + 1, & x > 2 \end{cases}$$

**20.5 FINDING LIMITS OF SOME OF THE IMPORTANT FUNCTIONS**
(i) Prove that  $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$  where n is a positive integer.



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## ANSWERS



Notes

## CHECK YOUR PROGRESS 20.1

1. (a) 17 (b) 7 (c) 0 (d) 2  
(e) -4 (f) 8
2. (a) 0 (b)  $\frac{3}{2}$  (c)  $-\frac{2}{11}$  (d)  $\frac{q}{b}$  (e) 6  
(f) -10 (g) 3 (h) 2
3. (a) 3 (b)  $\frac{7}{2}$  (c) 4 (d)  $\frac{1}{2}$
4. (a)  $\frac{1}{2}$  (b)  $\frac{1}{2\sqrt{2}}$  (c)  $\frac{1}{2\sqrt{6}}$  (d) 2 (e) -1
5. (a) Does not exist (b) Does not exist
6. (a) 0 (b)  $\frac{1}{4}$  (c) does not exist
7. (a) 1, -2 (b) 1 (c) 19
8. a = -2
10. limit does not exist

## CHECK YOUR PROGRESS 20.2

1. (a) 2 (b)  $\frac{e^2 - 1}{e^2 + 1}$
2. (a)  $-\frac{1}{e}$  (b) -e
3. (a) 2 (b)  $\frac{1}{5}$  (c) 0 (d)  $\frac{a}{b}$
4. (a)  $\frac{1}{2}$  (b) 0 (c) 4 (d)  $\frac{2}{3}$
5. (a)  $\frac{a^2}{b^2}$  (b) 2 (c)  $\frac{1}{2}$
6. (a) 1 (b)  $\frac{\pi}{2}$  (c) 0
7. (a)  $\frac{5}{3}$  (b)  $\frac{7}{4}$  (c) -5